CURVES and FLATS
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Background: Raising a Pattern, Keeping a Sheet Flat

What is the subject here? The aim is to gain control of this medium, mostly so as to be able to make those faces, my main proving ground. And the medium itself—is what? Clearly it involves folding curves (in, as it happens, rectangles of brown wrapping paper spray-glued to thick aluminum foil). Now curve-folding, as is known, creates surfaces that won’t lie flush to each other, or “open folds”; and open folds can be made voluntarily with straight lines too. So maybe our subject is best described as the Manipulation of Open Folds, whether curved or not. And gaining control of this subject, taming it, means, for me—as in certain political theories—flattening it: being able to crush, squeeze, twist, bend the thing to the right or left…. In short, I want to be able to restore an average flatness to a surface deformed by curved or open folds, and then see whether and how such a textured or raised surface can be further manipulated.

But let’s start at the beginning. Suppose you put curved folds of any kind in a flat sheet of paper. The paper will no longer lay flat. For that matter, you can use straight folds to create a curving surface easily enough—a cone for example—by making an angled crimp (mountain-valley pair) that originates in the interior of the sheet. But with non-flat surfaces made only via straight-line folds, you can [a] always collapse the surface to a completely flat state (while retaining the initial folds) via a finite number of additional straight-line folds. Curved surfaces made by curved folds cannot [b] be collapsed flat while retaining those folds by any finite means, neither by adding straight nor by adding curved folds. [Both of these conjectures seem to me eminently provable.]

If real flatness is not to be had, there is still the next-best thing, average flatness. Here the surface has a raised texture of essentially the same height and depth
throughout. The surface gives some of the appearance of flatness and shares some of its properties. This article mainly addresses some of the issues involved in making and manipulating surfaces of this kind.

For a surface with a curved fold to be kept flat on average, a pattern of curves of roughly similar shape will typically be drawn on it. This can be done in one direction, with curves (e.g., waves) running parallel to each other. Can it be done in two? Clearly it can, for one instinctively flattens a cone-shape (a surface created by a flat-fold) by means of concentric circles. But another, less explored possibility is to divide a flat surface into a lattice of squares, triangles or hexagons, and to place the identical curve-pattern in each. This has the nice effect of shrinking the paper by the same amount in all places, so it is not forced to bend from the plane. And it not only maintains average flatness, but also yields a surface that is similar in outline to the one that was started with.

However: this trick can’t be done with every pattern, only with those that line up or tessellate—so that the left line in one tile’s pattern turns into a right line in the tile next to it, and ditto for tops and bottoms.

**Spiral Curvigami Tessellations**

One ancient, well-studied pattern that tessellates very nicely is the vortex or spiral, so I want to spend a little time on it.

A vortex, whether in a bathtub or a galaxy, is nature’s way of pulling material in a plane toward a focus in the least objectionable manner, so it’s an intuitive choice for shrinking a sheet of paper too. Liquid swirls were investigated observationally by Leonardo da Vinci [19]; and patterns of ornamental spirals are to be found in the decorative art of many ancient cultures.

In origami too, spirals and vortex-like twist-folds have a distinguished pedigree, having been studied by, among others, S. Fujimoto, T. Kawasaki, Alex Bateman, Tomoko Fuse, Jeremy Schafer, Chris Palmer—indeed it sometimes seems by all the pioneers of the currently exploding field of origami tessellations. The spiral tessellations I’m introducing here are necessarily related to some of those more familiar ones in their underlying geometry, and they have other points in common too. But one difference is that the spirals here are drawn on the surface as curves and then folded directly—causing the paper to condense—rather than being created from straight folds of relatively free material in already condensed paper, folds which are then twisted into spirals. These spiral patterns are, for all that, one type of origami tessellation: they belong to the subset of tessellations which can be formed continuously with a lateral, bi-directional
With a square and triangular grid, you can make spirals that curve in the same direction [6], or you can alternate the direction [7]; the pattern will still line up. (With a hexagonal grid more thought is needed to achieve alternation.) Figure [3] is from a triangular grid with a uni-directional spiral; figures [12] and [13] use a square grid. (In these patterns the eye is naturally drawn to the shaded hollows between the ridges, but if you look at the vertices you’ll see the pattern’s spiral basis.) Figure [4] is a fancy version of a spiral pattern based on a hexagonal grid; [10] and [11] use simple spirals, also from hexagons.

Interestingly, an alternating spiral pattern compresses inward from the sides much less than a uni-directional pattern does. The degree of lateral compressibility is an important issue for any open-fold pattern, although it takes some practice to be able to recognize from a pattern drawing alone how well it will compress. (I won’t dwell on this subject here, but the issue of tangents, touched on below, bears on compressibility.) It should be remembered that when a pattern contains curves it will not compress all the way, in the nice way that a Miura-fold does.

Note too that a regular division of the plane is not necessary for shrinking a sheet via spirals: any irregular polygonal lattice will do. Figure [8] shows a surface carved at random into irregular polygons, and [9] a (semi-regular) spiral crease pattern for it. It is trivial to prove that any division into regular or irregular polygons will allow a spiral pattern to be created for it, and that
the pattern will fold. It is less trivial to prove that a surface so divided and folded can always be made to lay flat—for the possible reason that this may not be true. In my own experiments, since the spiral within each polygon can be twisted with a certain independence from its neighbors, one always has some control over how flat or curved the overall surface will be. On the other hand, when the polygons are of a different size and the spirals in them are of a different height the concept of ‘average flatness’ loses some of its clarity.

It is also worth suggesting that these individually-tweakable spirals may form a two-dimensional analog of sorts to the one-dimensional case shown by Eric Demaine in his talk at 4OSME. In the process of converting one outline (e.g., of a maple leaf) to another (an airplane), sometimes a segment of the outline would rotate around itself and thus spool in or spool out segments, adding or subtracting outline length. Something similar can be done with these open-fold spirals, shrinking or expanding the horizontal aspect of individual regions, though at the considerable cost of ceding flatness.

As an autobiographical note, I came upon this subject of spiral tessellations only when seeking an elegant solution to a sculptural problem that was nagging me: how to make from paper the dome of a person’s head, which curves in two directions at once, as paper is loth to do. Many curve-based tessellations, while they can be kept flat, also introduce some bi-directional flexibility to a sheet of paper. Spiral ones also happen fortuitously to look like hair….

Finally, to put this discussion of spirals back into perspective: spiral tessellations are just one kind of open-fold tessellation that will shrink a surface while preserving average flatness. There are many others (e.g., [2]). Surfaces can also be shrunk without any tessellation at all using semi-regular [1] or random-crumple methods; and if edge proportions are allowed to change a great many other options are available. It
seems that this field of compressive, ‘flatness’-preserving deformations of a sheet is still wide open for exploration in origami.

**Folding Patterned Sheets**

Let’s move to our other main area of investigation. Once you have a surface with a raised pattern on it, what can you do with it? Specifically, can the usual origami manipulations done on smooth sheets be done on these textured ones too?

The answer, I’m afraid, is usually no: most elaborate origami folding will typically be interfered with by the existence of a raised pattern. A counter-example among top-rank models is Roman Diaz’ Tiger’s Head [14: his design, my fold]; but there the curves are put in at the final stage on the free flat edges which remain at that stage. Starting from the outset with a 3-D texture poses considerable difficulties for much origami. Having said that, folding a raised and especially a curved pattern around a corner-line can create deep furrows and bulges that are visually quite arresting—enough by itself to make a fine model, as the beautiful Tower-form made (1976) by the great pioneer of curved folding, David Huffman, clearly demonstrates [15]. Here, although the resultant shape has struck many people as wondrously complex, its crease pattern is actually quite simple [16; my reconstruction]. I have tried absorbing some of its principles in my own work [17].

The Huffman Tower, by the way, prompts a question that comes up more
generally from various quarters when dealing with curved folds: is there any difference of principle between a curved fold and a straight one? Isn’t a sine-curve just a zig-zag with the corners rounded off? In the case of the Huffman Tower, couldn’t all the curvy lines have been replaced with straight segments, and the curving surfaces with flat ones? (And how about with my spirals?) This is not an insignificant question, and while the answer may be different in each separate instance it is always worth asking. There are some real differences between curved and straight folding, but the effect of curves can also be so hypnotic as to make us forget to check whether straight-line analogs exist. But let us leave that aside for now.

I want to consider what happens when a surface that is patterned in the way I’ve been describing is folded along a line—folded gradually anywhere from zero to 180 degrees. There are four different types of simple encounters of open folds (for now: mountain-valley pairs) with a corner line, and I’d like to show what happens in each.

Figure [18] is an open-fold crease pattern, in which you are to imagine (or attempt) folding the more horizontal lines first into open mountain and valley folds, and then bending the pattern successively at each of the four vertical locations.

If you try bending the straight-line open-folds at A, the paper will resist. Eventually it will buckle, that is, form new fold-lines at awkward and unexpected locations. This is the corrugation effect, used for adding stability to flimsy sheet materials. Note that since the lines that intersect A are all straight, there is nothing stopping you from folding them all the way into closed-folds; A can then be folded without complaint.

At B, the ‘horizontal’ open-fold lines, which are shown to be straight but may also be curved, meet line B from both sides at an angle. (Line B in fact will already be formed by having made the angled open-folds.) Bending the surface here can be done quite easily: the corrugation effect has disappeared. However, the result of such bending is that the height of the surface will compress along B, as the angles turn inward and trade some of their verticality for depth. If the open
folds meeting B are straight-lines, a 180° bend around B will close these folds completely.

At C, the horizontal lines are arcs; a hard fold along C itself encounters the same resistance as at A and for the same reasons. However, the region of C taken as a whole behaves just as the single line of B does; in fact it can be considered a stretched out version of B (one dimension stretching into two!). Thus the entire region of C can be made into a corner that curves gradually, and if the corner is sharpened (edges bent back more), the furrows will deepen just as they did at B. The height will likewise shrink. But because they are curved the folds will never shut completely. (On the other hand you are able to bend the surface back by more than 180°, indeed by more than 360°.)

At D, the open folds meet the line at a tangent: an angle of zero. Consequently there are no angles to rotate inward, and a fold here leaves the vertical extension unchanged (though there may be bending at the tangent point). It may be noted that this property of being able to meet a line at a tangent is one which curves possess and straight segments do not, so this is yet another answer to the question what differences there are for folding purposes between curves and straights.

None of the above is earth-shaking mathematics, but it does account for many of the simpler cases of raised-pattern folding, so it needs to be stated. Fancier permutations (non-parallel mountains and valleys, mountain + mountain + valley open folds, open folds that meet curves, etc.) are of course possible too.

**Concluding Thoughts**

I think this is enough of a sketch to suggest some of the issues that come up when forming and manipulating curve-patterns. I want to conclude with a few thoughts about method and the links and tensions here between art and science.

For experimental work, the ideal medium for curved folding is a foil-backed paper (preferably stiff foil, 50 – 100 microns thick) rather than paper on which the pattern has been plotted and scored. The reason is not aesthetic—aesthetics may in fact favor plain paper—but rather that foil-paper, which holds a curvy shape without springing, also allows you to erase a line with a fingernail and shift your curve at will. This helps avoid what I think is the main mistake that’s affected some of the people who have written about curved folds: the assumption that if a curve representing a particular mathematical function creates a nice effect, the effect is due to the function and no other curve can accomplish about the same thing. You can avoid such fixation by trying out other curves and straight-line variants—but that requires a comfortable medium for doing this. (In my opinion
this mistake has also led to a wrongly mathematicized concept about what curved folding involves: there is a math here—I may have been hinting at it—but it would seem to be more ‘topological’ than algebraic.)

A similar fixation tends to happen with regular patterns, so these should always be tested against the most irregular version of the same pattern to see what in fact is doing the work.

Irregularity vs. regularity, plotted and repeated patterns vs. freeform and varying curves—all this raises another issue, this time a purely aesthetic one: the old, grand tension between mathematical optima and repeatability on the one side, and romantic and individual expression on the other. This is rather a large topic to broach right here: entire cultures are defined by just where and how they come out on this continuum. I will say only this. Certainly in the animal world, the outline curve is a prime bearer of information about a living form’s identity and emotional state; and in the handwriting and drawing of humans, the curve or the flourish is where personality is looked for—and found. It would be a shame if origami’s inherent tendency for pattern and repetition should give rise in this new field too to mainly a cold and crystalline form of model design, to the calculated and repeated rather than the expressive. Curved folds leave a great deal of freedom for the shaping of 3D form: too much of it, to many folders’ tastes. But where there is freedom there can also be—individuality.

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